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Abstract

The stochastic calculus for Gaussian processes is applied to obtain a Tanaka formula for a Volterra-type multifractional Gaussian process. The existence and the regularity properties of the local time of this process is obtained by means of Berman's Fourier analytic approach.

1 Introduction

Several types of multifractional Gaussian processes have been studied, including the processes introduced by Lévy-Véhel and Peltier [11] and by Benassi, Jaffard and Roux [6], known as the moving average and the harmonisable versions of multifractional Brownian motion, and the processes introduced by Benassi, Cohen and Istas [2] and, based on the calculus of (multi-)fractional differentiation and integration, by Surgailis [17]. These processes are usually defined by replacing, in certain representations of fractional Brownian motion (fbm), the Hurst parameter H by a Hurst function h, i.e. a real function of the time parameter with values in (0,1). Generalisations to stable processes have been studied too [10, 15, 16]. The stochastic properties and the regularity of the trajectories of these processes can be characterised in the same terms as for fbm, in particular by the property of local asymptotic self-similarity or by the pointwise Hölder exponent, which is constant and equal to H for fbm, but varies in time following h in the multifractional case.

The aim of this article is to study a Volterra type multifractional Gaussian process B_h which fits into the framework of stochastic calculus, and its local time. In fact, the stochastic calculus, including the Itô formula and stochastic differential equations, is now well established for fbm, where stochastic integrals have been defined in the Malliavin sense or by means of Wick products. The stochastic calculus for multifractional Gaussian processes has not yet been developed explicitely, but important elements, including a divergence integral and an Itô formula, have been proven in [1] for a class of Gaussian processes admitting a kernel representation with respect to Brownian motion. The process B_h we study in this article is defined by allowing for a Hurst function in the kernel representation of fbm, and it belongs to the class of Gaussian processes studied in [1]. The Itô formula in [1] is applied to get a representation of the local time for which, as it is usually the case in Tanaka-like formulas, the occupation measure is not the Lebesgue measure, but the quadratic variation process. However, contrary to fbm, the interpretation in our case is more delicate because the quadratic variation process is not necessarily increasing (but of bounded variation). We compare this local time to the local time with respect to the Lebesgue measure, which we obtain, together with the regularity properties of its trajectories, by the Fourier analytic approach initiated by S. Berman [3] and which is based on the notion of local nondeterminism (LND). Sufficient conditions for the LND property to hold are given in [7], and we show that these conditions are implied by the stochastic properties, in particular the lass property, of B_h.

It is well known that the cases where the Hurst parameter or the Hurst function is $< 1/2$ resp. $> 1/2$ have to be treated separately (see [1] and [12]). We are interested in the case where the Hurst function h takes values in (1/2,1). In fact, the Volterra-type process we study here is defined only for this case, which is appropriate for long memory applications [14]. The article is organised as follows : In Section 2 stochastic properties and the regularity of the trajectories of B_h are proven. The form of the covariance function (Proposition 2) shows in particular that B_h differs from the harmonisable and moving average multifractional Brownian motions. The continuity of the trajectories of B_h is obtained by classical criteria from estimates for the second order moments of the increments (Proposition 3). The lass property is proven in Proposition 5; it implies that the pointwise Hölder exponent of B_h is equal to h (Proposition 6). In Section 3 B_h is shown to satisfy the property named (K4) in [1] and that the quadratic variation of B_h is of bounded variation. Therefore the divergence integral and the Itô formula developed in [1] (recalled briefly in this section) hold for B_h. Section 4 is devoted to the local time of B_h. Its existence and square integrability with respect to the space variable follows from a classical criterion due to S. Berman ([3]) (Proposition 11). Then a Tanaka-type formula with the divergence integral of Section 3 is given (Theorem 12), where the occupation measure of the local time is the quadratic variation of B_h. Finally the local time with respect to the Lebesgue measure and its regularity in the space and time variables are given in Theorem 16. The proofs are based on the LND property, which is shown to hold for B_h in Proposition 14. The sufficient conditions, named A and A_m are in fact closely related to the lass property, which is the main ingredient for the proof of Proposition 14. Proposition 15 shows that the condition A holds in fact for a much larger class of Gaussian processes than B_h.

2 A Volterra-type multifractional Gaussian process

It is well-known that the fractional Brownian motion B_H with (fixed) Hurst parameter $H \in (1/2, 1)$ can be represented for any $t \geq 0$ as

$$B_H(t) \quad = \quad \int_0^t K_H(t,u) W(du),$$

where

$$K_H(t,u) = c_H u^{1/2-H} \int_u^t (y-u)^{H-3/2} y^{H-1/2} dy,$$

with

$$c_H \quad = \quad \left(\frac{\pi H(2H-1)}{\Gamma(2-2H)\Gamma(H+1/2)^2 \sin(\pi(H-1/2))} \right)^{1/2} (H-1/2)$$

and $W(dy)$ is a Gaussian measure.

Let a and b be two real numbers satisfying $1/2 < a < b < 1$. Throughout the paper we consider a function $h : \mathbb{R} \to [a,b]$. We assume that this function is $\beta-$Hölder with $\sup h < \beta$. Define the centered Gaussian process $B_h = \{B_h(t), t \geq 0\}$ by

$$B_h(t) = B_{h(t)}(t) = \int_0^t K_{h(t)}(t,u) W(du),$$

where

$$K_{h(t)}(t,u) = u^{1/2-h(t)} \int_u^t (y-u)^{h(t)-3/2} y^{h(t)-1/2} dy. \tag{1}$$

If $h(\cdot) = H$ is a constant, B_H is a fractional Brownian motion up to the multiplicative constant c_H. Before establishing properties of B_h we give a lemma regarding an estimate on K_H that we use throughout the paper.

Lemma 1 *For every $T > 0$ there exists a function $\Phi_T \in L^2((0,T], \mathbb{R}_+)$ such that for every $s \in (0,T]$*

$$\sup_{\lambda \in [a,b], t \in (0,T]} \left| \frac{\partial}{\partial \lambda} K_\lambda(t,s) \right| \leq \Phi_T(s).$$

Proof. We have

$$\begin{aligned} \frac{\partial}{\partial \lambda} K_\lambda(t,s) \quad &= \quad (-\log s) s^{1/2-\lambda} \int_s^t (y-s)^{\lambda-3/2} y^{\lambda-1/2} dy \\ &\quad + s^{1/2-\lambda} \int_s^t (y-s)^{\lambda-3/2} y^{\lambda-1/2} (\log(y-s) + \log y) dy. \end{aligned}$$

Then for every $T > 0$ there exists a constant $C_{a,b,T}$ such that for every $\lambda \in [a,b]$ and $s \in (0,T]$

$$\left| \frac{\partial}{\partial \lambda} K_\lambda(t,s) \right| \leq C_{a,b} (1 \vee |\log s|) s^{1/2-b} =: \Phi_T(s)$$

that concludes the proof. ∎

The following proposition gives the covariance of this process.

Proposition 2 *Let $X = \{X(t,\lambda), t \geq 0, \lambda \in (1/2,1)\}$ be the two-parameter process given by $X(t,\lambda) = \int_0^t K_\lambda(t,u) W(du)$. Then*

$$\mathbb{E}[X(t,\lambda) X(s,\lambda')] = \int_0^t dy \int_0^s dz \widetilde{\beta}(y,z,\lambda,\lambda') |y-z|^{\lambda+\lambda'-2} \left(\frac{y}{z} \right)^{\lambda-\lambda'}$$

where

$$\widetilde{\beta}(y,z,\lambda,\lambda') = \beta(2-\lambda-\lambda', \lambda'-1/2) 1_{\{y>z\}} + \beta(2-\lambda-\lambda', \lambda-1/2) 1_{\{y<z\}}$$

and $\beta(a,b)$ $(a,b>0)$ is the Beta function. In particular,

$$\begin{aligned}
\mathbb{E}[X(t,\lambda)^2] &= \int_0^t dy \int_0^t dz\ \beta(2-2\lambda,\lambda-1/2)|y-z|^{2\lambda-2} \\
&= \frac{\beta(2-2\lambda,\lambda-1/2)}{\lambda(2\lambda-1)}\ t^{2\lambda}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\mathbb{E}[X(t,\lambda)X(s,\lambda')] &= \int_0^{t\wedge s} K_\lambda(t,u)K_{\lambda'}(s,u)du \\
&= \int_0^{t\wedge s} u^{1-\lambda-\lambda'}\left(\int_u^t (y-u)^{\lambda-3/2}y^{\lambda-1/2}dy\right) \\
&\qquad \times\left(\int_u^s (z-u)^{\lambda'-3/2}z^{\lambda'-1/2}dz\right)du \\
&= \int_0^t dy\int_0^s dz\, y^{\lambda-1/2}z^{\lambda'-1/2} \\
&\qquad \times\int_0^{y\wedge z} u^{1-\lambda-\lambda'}(y-u)^{\lambda-3/2}(z-u)^{\lambda'-3/2}du.
\end{aligned}$$

We fix $y>z$ and calculate the following integral by making the successive substitutions $u=vz$, $w=(y-vz)/(1-v)$, $t=w/y$ and $s=1/t$:

$$\begin{aligned}
&\int_0^z u^{1-\lambda-\lambda'}(y-u)^{\lambda-3/2}(z-u)^{\lambda'-3/2}du \\
&\quad = z^{1/2-\lambda}\int_0^1 \lambda'(y-vz)^{\lambda-3/2}(1-v)^{\lambda'-3/2}dv \\
&\quad = z^{1/2-\lambda}(y-z)^{\lambda+\lambda'-2}\int_y^{+\infty}(w-y)^{1-\lambda-\lambda'}w^{\lambda-3/2}(y-z)^{\lambda+\lambda'-2}dw \\
&\quad = z^{1/2-\lambda}y^{1/2-\lambda'}(y-z)^{\lambda+\lambda'-2}\int_1^{+\infty}(t-1)^{1-\lambda-\lambda'}t^{\lambda-3/2}dt \\
&\quad = z^{1/2-\lambda}y^{1/2-\lambda'}(y-z)^{\lambda+\lambda'-2}\beta(2-\lambda-\lambda',\lambda'-1/2).
\end{aligned}$$

In the same way we get for $y<z$:

$$\begin{aligned}
&\int_0^y u^{1-\lambda-\lambda'}(y-u)^{\lambda-3/2}(z-u)^{\lambda'-3/2}du \\
&\quad = z^{1/2-\lambda}y^{\frac{1}{2}-\lambda'}(z-y)^{\lambda+\lambda'-2}\beta(2-\lambda-\lambda',\lambda-1/2).
\end{aligned}$$

This concludes the proof. ■

In the sequel we need the estimates we establish in the following proposition.

Proposition 3 *The process X satisfies the following estimates.*
a) For all s and $t\geq 0$

$$\mathbb{E}[(X(t,\lambda)-X(s,\lambda))^2] = c_\lambda^{-2}|t-s|^{2\lambda} \tag{2}$$

b) For every $T>0$, there exists a constant $C_T>0$ such that for every $t\in[0,T]$ and every λ and $\lambda'\in[a,b]$

$$\mathbb{E}[(X(t,\lambda)-X(t,\lambda'))^2] \leq C_T|\lambda-\lambda'|^2 \tag{3}$$

Proof.
Proof of a). For every $\lambda \in [a,b]$ the process $X(\cdot,\lambda) : t \mapsto X(t,\lambda)$ is a fractional Brownian motion with variance c_λ^{-2} so we deduce (2).
Proof of b). We have

$$\mathbb{E}[(X(t,\lambda) - X(t,\lambda'))^2] = \int_0^t (K_\lambda(t,u) - K_{\lambda'}(t,u))^2 du.$$

There exists $\xi = \xi(\lambda,\lambda') \in [\min\{\lambda,\lambda'\}, \max\{\lambda,\lambda'\}]$ such that

$$K_\lambda(t,u) - K_{\lambda'}(t,u) = (\lambda - \lambda') \left| \frac{\partial}{\partial \lambda} K_\lambda(t,u) \right|_{\lambda=\xi=\xi(\lambda,\lambda')}.$$

Then, thanks to Lemma 1 we get for every t, λ and λ'

$$\mathbb{E}[(X(t,\lambda) - X(t,\lambda'))^2] \leq |\lambda - \lambda'|^2 \int_0^T (\Phi_T(u))^2 du$$

that ends the proof. ■

From the last proposition we can deduce the continuity of B_h.

Corollary 4 *The process B_h defined above has continuous trajectories.*

Proof. We deduce from Proposition 3 that for every s and t in a compact interval $[0,T]$ such that $|t-s| < 1$

$$\begin{aligned}
\mathbb{E}[(B_h(t) - B_h(s))^2] &= \mathbb{E}[(X(t,h(t)) - X(s,h(s)))^2] \\
&\leq 2\mathbb{E}[(X(t,h(t)) - X(t,h(s)))^2] \\
&\quad +2\mathbb{E}[(X(t,h(s)) - X(s,h(s)))^2] \\
&\leq 2C_T|h(t) - h(s)|^2 + 2c_{h(s)}^{-2}|t-s|^{2h(s)} \\
&\leq 2C_T|t-s|^{2\beta} + 2 \sup_{\lambda \in [a,b]} (c_\lambda^{-2})|t-s|^{2a}.
\end{aligned}$$

Then $\mathbb{E}[(B_h(t) - B_h(s))^2]/|t-s|^{2a}$ is bounded and since B_h is Gaussian we can deduce from [5] its continuity. ■

Now we deal with the local self-similarity property of B_h.

Proposition 5 *The process B_h is locally self-similar. More precisely, for every t, we have the following convergence in distribution:*

$$\lim_{\varepsilon \to 0} \left(\frac{B_h(t+\varepsilon u) - B_h(t)}{\varepsilon^{h(t)}} \right)_{u \geq 0} = \left(c_{h(t)}^{-1} B_{h(t)}(u) \right)_{u \geq 0}$$

where $\lim_{\varepsilon \to 0}$ stands for the limit in distribution in the space of continuous functions endowed with the uniform norm on every compact set.

Proof. Let us start by proving the convergence of the finite dimensional distribution. Because B_h is Gaussian, it suffices to show the convergence of the second-order moments. We then can write for every u and v:

$$\begin{aligned}
&\frac{1}{\varepsilon^{2h(t)}} \mathbb{E}\left[(B_h(t+\varepsilon u) - B_h(t)) (B_h(t+\varepsilon v) - B_h(t)) \right] \\
&\qquad = \frac{1}{\varepsilon^{2h(t)}} (I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon) + I_4(\varepsilon))
\end{aligned}$$

where

$$\begin{aligned}
I_1(\varepsilon) &= \mathbb{E}[(X(t+\varepsilon u, h(t)) - X(t, h(t))) \\
&\qquad \times (X(t+\varepsilon v, h(t)) - X(t, h(t)))], \\
I_2(\varepsilon) &= \mathbb{E}[(X(t+\varepsilon u, h(t+\varepsilon u)) - X(t+\varepsilon u, h(t))) \\
&\qquad \times (X(t+\varepsilon v, h(t)) - X(t, h(t)))], \\
I_3(\varepsilon) &= \mathbb{E}[(X(t+\varepsilon u, h(t)) - X(t, h(t))) \\
&\qquad \times (X(t+\varepsilon v, h(t+\varepsilon v)) - X(t+\varepsilon v, h(t)))], \\
I_4(\varepsilon) &= \mathbb{E}[(X(t+\varepsilon u, h(t+\varepsilon u)) - X(t+\varepsilon u, h(t))) \\
&\qquad \times (X(t+\varepsilon v, h(t+\varepsilon v)) - X(t+\varepsilon v, h(t)))].
\end{aligned}$$

Thanks to the selfsimilarity of the fractional Brownian motion and by stationarity of its increments we get

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{2h(t)}} I_1(\varepsilon) = \frac{1}{2c_{h(t)}^2}(|u|^{2h(t)} + |v|^{2h(t)} - |u-v|^{2h(t)}). \tag{4}$$

Then, because of Cauchy-Schwartz inequality, it is enough to prove

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{2h(t)}} \mathbb{E}\left[(X(t+\varepsilon u, h(t+\varepsilon u)) - X(t+\varepsilon u, h(t)))^2\right] = 0 \tag{5}$$

to get

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{2h(t)}} (I_2(\varepsilon) + I_3(\varepsilon) + I_4(\varepsilon)) = 0. \tag{6}$$

So, using Lemma 1 we get

$$\begin{aligned}
&\frac{1}{\varepsilon^{2h(t)}} \mathbb{E}\left[(X(t+\varepsilon u, h(t+\varepsilon u)) - X(t+\varepsilon u, h(t)))^2\right] \\
&\qquad = \frac{1}{\varepsilon^{2h(t)}} \int_0^{t+\varepsilon u} \left(K_{h(t+\varepsilon u)}(t+\varepsilon u, s) - K_{h(t)}(t+\varepsilon u, s)\right)^2 ds \\
&\qquad \le \frac{(h(t+\varepsilon u) - h(t))^2}{\varepsilon^{2h(t)}} \int_0^{t+\varepsilon u} \sup_H \left|\frac{\partial}{\partial H} K_H(t+\varepsilon u, s)\right|^2 ds \\
&\qquad \le C\varepsilon^{2\beta - 2h(t)} \int_0^{t+u} |\Phi_{t+u}(s)|^2 \, ds \underset{\varepsilon\to 0}{\longrightarrow} 0.
\end{aligned}$$

Now it remains to prove the tightness in the space of continuous functions endowed by the uniform norm. We also consider $T > 0$ such that t, $t+\varepsilon u$ and $t+\varepsilon v \in [0, T]$ for all ε. Making similar calculations as in the proof of Corollary 4 we get that there exist $C_T > 0$ such that

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{B_h(t+\varepsilon u) - B_h(t)}{\varepsilon^{h(t)}} - \frac{B_h(t+\varepsilon v) - B_h(t)}{\varepsilon^{h(t)}}\right)^2\right] \\
&\qquad = \frac{1}{\varepsilon^{2h(t)}} \mathbb{E}\left[(B_h(t+\varepsilon u) - B_h(t+\varepsilon v))^2\right] \\
&\qquad \le \frac{C_T}{\varepsilon^{2h(t)}} |\varepsilon u - \varepsilon v|^{2h(t+\varepsilon v)} \\
&\qquad = C_T \varepsilon^{2(h(t+\varepsilon v) - h(t))} |u-v|^{2h(t+\varepsilon v)}
\end{aligned}$$

Since h is $\beta-$Hölder then $\varepsilon^{2(h(t+\varepsilon v)-h(t))}$ is uniformly bounded. Moreover $h(t+\varepsilon v) \geq a$, thus

$$\mathbb{E}\left[\left(\frac{B_h(t+\varepsilon u)-B_h(t)}{\varepsilon^{h(t)}}-\frac{B_h(t+\varepsilon v)-B_h(t)}{\varepsilon^{h(t)}}\right)^2\right] \leq C_T\,|u-v|^{2a}$$

that ends the proof. ■

It is classical to deduce pointwise Hölder continuity from local self-similarity [2]. We recall that the pointwise Hölder continuity of a function f is characterized by the pointwise Hölder exponent $\alpha_f(t_0)$ defined at each point t_0 as

$$\alpha_f(t_0) = \sup\left\{\alpha > 0 : \lim_{t\to t_0} \frac{|f(t)-f(t_0)|}{|t-t_0|^\alpha} = 0\right\}.$$

Proposition 6 *For every $t_0 \in \mathbb{R}_+$ the pointwise Hölder exponent $\alpha_{B_h}(t_0)$ of B_h is almost surely equal to $h(t_0)$.*

Proof. We deduce from Proposition 5 that $\alpha_{B_h}(t_0) \leq h(t_0)$. Now we prove that $\alpha_{B_h}(t_0) \geq h(t_0)$. Let $\varepsilon > 0$. For every s and $t \in [t_0-\varepsilon, t_0+\varepsilon]$ such that $|t-s| < 1$, from Proposition 3 we have

$$\mathbb{E}[(B_h(t)-B_h(s))^2] \quad \leq \quad C|t-s|^{2\inf_{[t_0-\varepsilon,t_0+\varepsilon]} h}.$$

By the fact that B_h is Gaussian and applying Kolmogorov theorem [5], we get that $\lim_{t\to t_0} |f(t)-f(t_0)|/|t-t_0|^\alpha = 0$ for every $\alpha < \inf_{[t_0-\varepsilon,t_0+\varepsilon]} h$. This holds for every $\varepsilon > 0$, so by continuity of h we get $\lim_{t\to t_0} |f(t)-f(t_0)|/|t-t_0|^\alpha = 0$ for every $\alpha < h(t_0)$. We can deduce that $\alpha_{B_h}(t_0) \geq h(t_0)$, and hence $\alpha_{B_h}(t_0) = h(t_0)$ ■

3 Stochastic calculus for B_h

The aim of this section is to apply the stochastic calculus developed by Alòs, Mazet and Nualart in [1] to get a stochastic integral for B_h and an Itô formula. We recall that in [1] the following hypothesis, called ($\mathbb{K}4$), appears for regular kernels:

- **Hypothesis** ($\mathbb{K}4$). For all $s \in [0,T)$, $K(\cdot,s)$ has bounded variation on the interval $(s,T]$, and $\int_0^T |K|((s,T],s)^2 ds < \infty$.

Lemma 7 *Suppose that h is of bounded variation on $(s,T]$ for all $s \in [0,T)$. Then ($\mathbb{K}4$) holds for $(t,s) \mapsto K_{h(t)}(t,s)$ defined by (1).*

Proof. Let

$$\mathrm{Var}^n_{(s,T]}(\cdot,s) = \sup_{t_0=s<t_1<\ldots<t_n=T} \sum_{i=1}^n |K_{h(t_i)}(t_i,s) - K_{h(t_{i-1})}(t_{i-1},s)|,$$

and suppose without restriction of generality that $T=1$. Then

$$|K_{h(t_i)}(t_i,s) - K_{h(t_{i-1})}(t_{i-1},s)| \quad \leq I_1(i) + I_2(i),$$

where

$$I_1(i) = |K_{h(t_i)}(t_i,s) - K_{h(t_i)}(t_{i-1},s)|$$

and

$$I_2(i) = |K_{h(t_i)}(t_{i-1}, s) - K_{h(t_{i-1})}(t_{i-1}, s)|$$

We have

$$\begin{aligned} I_1(i) &= s^{1/2-h(t_i)} \int_{t_{i-1}}^{t_i} (y-s)^{h(t_i)-3/2} y^{h(t_i)-1/2} dy \\ &\leq s^{1/2-b} \int_{t_{i-1}}^{t_i} (y-s)^{a-3/2} y^{a-1/2} dy. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i=1}^{n} |K_{h(t_i)}(t_i, s) - K_{h(t_i)}(t_{i-1}, s)| &\leq s^{1/2-b} \int_s^1 (y-s)^{a-3/2} y^{a-1/2} dy \\ &\leq C(a) s^{1/2-b}. \end{aligned} \tag{7}$$

Regarding I_2 we have

$$\begin{aligned} &\sum_{i=1}^{n} \left|K_{h(t_i)}(t_{i-1}, s) - K_{h(t_{i-1})}(t_{i-1}, s)\right| \\ &\qquad \leq \sum_{i=1}^{n} |h(t_i) - h(t_{i-1})| \sup_{s \leq t \leq 1, a \leq \lambda \leq b} \left| \frac{\partial}{\partial \lambda} K_\lambda(t, s) \right|. \end{aligned}$$

The proof of Lemma 1 implies that $\mathrm{Var}^n_{(s,T]}(\cdot, s) \leq C(1 \vee \mid \log s \mid) s^{1/2-b}$, where the constant $C > 0$ depends on h (but not on n). This implies that $(\mathbb{K}4)$ holds for K_h. ■

Remark 8 *Since h is also supposed $\beta-$Hölder continuous for some $\beta \leq 1$, we will suppose from now on that h is Lipschitz-continuous. Notice that $\lim_{t \searrow u} K_{h(t)}(t, u) = 0$ for all $u > 0$.*

For simplicity we write K instead of K_h but keep in mind that $(t, s) \to K(t, s)$ means $(t, s) \to K_{h(t)}(t, s)$ and that differentials with respect to t act also via h.

In the sequel we need the following proposition regarding the variance of B_h.

Proposition 9 *The variance $s \mapsto R_s = \mathbb{E}[B_h(s)^2]$ is of bounded variation on $(0, T]$.*

Proof. We have

$$\begin{aligned} \sum_{i=1}^{n} \left|R_{s_{i+1}} - R_{s_i}\right| &= \sum_{i=1}^{n} \left| \int_0^{s_{i+1}} K(s_{i+1}, r)^2 dr - \int_0^{s_i} K(s_i, r)^2 dr \right| \\ &\leq \sum_{i=1}^{n} \int_0^{s_{i+1}} K(s_{i+1}, r)^2 dr + \sum_{i=1}^{n} \left| \int_0^{s_i} [K(s_{i+1}, r)^2 - K(s_i, r)^2] dr \right|. \end{aligned}$$

The functions $|K(s, r) 1_{[0,s]}(r)|$ are bounded by the square integrable function $k(r) = |K(T, r)| + |K|((r, T], r)$. Hence the first term above is upper bounded by $\int_0^T k(r)^2 dr$. The second term is upper bounded by

$$\begin{aligned} &\sum_{i=1}^{n} \int_0^{s_i} |K|((s_i, s_{i+1}], r) \, |K(s_{i+1}, r) + K(s_i, r)| \, dr \\ &\qquad \leq 2 \int_0^T k(r) |K|((r, T], r) dr < \infty. \end{aligned}$$

This concludes the proof. ■

For any $f \in L^2([0,T])$ let Kf be defined by $(Kf)(t) = \int_0^t K(t,s)f(s)ds$. Let $\mathcal{E}$ be the set of step functions on $[0,T]$, and let the operator K^* be defined on $\mathcal{E}$ by $(K^*\varphi)(s) = \int_s^T \varphi(t)K(dt,s)$. Then K^* is the adjoint of K. In fact, for $a_i \in \mathbb{R}$, $0 = s_1 < s_2 < ... < s_{n+1} = T$, $\varphi = \sum_{i=1}^n a_i 1_{(s_i,s_{i+1}]}(s)$ and $f \in L^2([0,T])$, we write

$$(K^*\varphi)(s) = \sum_{i=1}^n 1_{(s_i,s_{i+1}]}(s) \sum_{j=i}^n a_j (K(s_{j+1},s) - K(s_j,s))$$

and

$$\begin{aligned}
\int_0^T (K^*\varphi)(s)f(s)ds &= \sum_{j=1}^n a_j \int_0^T \sum_{i=1}^j 1_{(s_i,s_{i+1}]}(s)(K(s_{j+1},s) - K(s_j,s))f(s)ds \\
&= \sum_{j=1}^n a_j[(Kf)(s_{j+1}) - (Kf)(s_j)] = \int_0^T \varphi(t)(Kf)(dt). \qquad (8)
\end{aligned}$$

As usual the Reproducing Kernel Hilbert Space (RKHS) $\mathcal{H}$ is defined as the closure of the linear span of the indicator functions $\{1_{[0,t]}, t \in [0,T]\}$ with respect to the scalar product $\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = \mathbb{E}[B_{h(t)}(t)B_{h(s)}(s)] \equiv R(t,s)$. By replacing $f(s)ds$ in (8) by $W(ds)$, we have by (1)

$$B_h(\varphi) \equiv \int_0^T \varphi(t)B_h(dt) = \int_0^T (K^*\varphi)(s)W(ds).$$

Therefore

$$\|\varphi\|_{\mathcal{H}}^2 = \mathbb{E}[B_h(\varphi)^2] = \|K^*\varphi\|_{L^2([0,T])}^2 \leq \int_0^T \left[\int_0^T |\varphi(t)||K|(dt,s)\right]^2 ds =: \|\varphi\|_K^2.$$

Let us denote by $\mathcal{H}_K$ the completion of $\mathcal{E}$ with respect to the $\|\cdot\|_K$-norm. Then $\mathcal{H}_K$ is continuously embedded in $\mathcal{H}$.

In order to define the stochastic integral with respect to B_h, let us denote by $\mathcal{S}$ the set of smooth cylindrical random variables of the form $F = f(B_h(\varphi_1), B_h(\varphi_2), ..., B_h(\varphi_n))$, where $n \geqq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its derivatives are bounded) and $\varphi_1, \varphi_2, ..., \varphi_n \in \mathcal{H}$. Let us also denote by $\mathbb{D}^{1,2}(\mathcal{H}_K)$ the closure of $\{F \in \mathcal{S} : F \in L^2(\Omega, \mathcal{H}_K), DF \in L^2(\Omega \times \mathcal{H}_K, \mathcal{H}_K)\}$. Then $\mathbb{D}^{1,2}(\mathcal{H}_K)$ is included in the domain $\mathrm{Dom}(\delta^{B_h})$ of the divergence operator of B_h, and the integral of $u \in \mathrm{Dom}(\delta^{B_h})$ with respect to B_h is given by

$$\delta^{B_h}(u) \equiv \int_0^T u\, \delta B_h = \int_0^T (K^*u)\, \delta W \equiv \int_0^T \left[\int_s^T u(r)K(dr,s)\right] \delta W(s),$$

where the last two integrals are the divergence integrals with respect to Brownian motion. Let us recall that the integral $\delta^{B_h}(u)$ is defined, for any $u \in L^2(\Omega, \mathcal{H})$, as the unique element in $L^2(\Omega)$ which satisfies the duality relationship $\mathbb{E}(\delta^{B_h}(u)F) = \mathbb{E}\langle DF, u\rangle_{\mathcal{H}}$ for all $F \in \mathcal{S}$.

The following Itô formula, due to Alòs, Mazet and Nualart [1], will be applied in the next section. Let F be a function of class $C^2(\mathbb{R})$ satisfying the condition

$$\max\{|F(x)|, |F'(x)|, |F''(x)|\} \leq ce^{\lambda|x|^2},$$

where c and λ are positive constants such that $\lambda < \frac{1}{4}\left(\sup_{0\leqq t\leq T} R_t\right)^{-1}$. This implies that the process $F'(B_t)$ belongs to $\mathbb{D}^{1,2}(\mathcal{H}_K)$. The integral $\int_0^t F'(B_h(s))\delta B_h(s)$ is therefore well defined for all $t \in [0,T]$, and the following Itô-type formula holds ([1], Theorem 2):

$$F(B_h(t)) = F(0) + \int_0^t F'(B_h(s))\delta B_h(s) + \frac{1}{2}\int_0^t F''(B_h(s))dR(s). \tag{9}$$

4 Local time and Tanaka formula for B_h

First we prove by means of a criterion for Gaussian processes due to S. Berman that B_h has a local time with respect to the Lebesgue measure. Then we derive a Tanaka-type formula from the Itô formula of Section 3 and show that B_h satisfies the LND property. This implies continuity and Hölder regularities in space and in time of the trajectories of the local time.

Definition 10 *For any Borel set $C \subset \mathbb{R}_+$* the occupation measure m_C *of B_h on C is defined, for all Borel sets $A \subset \mathbb{R}$, by $m_C(A) = \lambda\{t \in C,\ B_{h(t)}(t) \in A\}$, where λ is the Lebesgue measure on $\mathbb{R}_+$. If m_C is absolutely continous with respect to the Lebesgue measure on $\mathbb{R}$, the* local time *(or occupation density) of B_h on C is defined as the Radon-Nikodym derivative of m_C and will be denoted by $\{L(C,x), x \in \mathbb{R}\}$. Sometimes we write $L(t,x)$ instead of $L([0,t],x)$.*

This definition implies that the local time of B_h satisfies the following occupation density formula

$$\int_C g(t, B_h(t))dt = \int_{C\times\mathbb{R}} g(t,x)L(dt,x)dx \tag{10}$$

for all continuous functions with compact support $g : C \times \mathbb{R} \to \mathbb{R}$. If g does not depend explicitely on t, we get the more classical occupation density formula where the right side of (10) is replaced by $\int_C g(x)L(C,x)dx$.

Proposition 11 *The local time of B_h exists P-a.s. on any interval $[0,T]$ and is a square integrable function of x.*

Proof. By [3], for any continuous and zero mean Gaussian process $\{X_t,\ t \in [0,T]\}$ with bounded covariance function, the condition

$$\int_0^T\int_0^T \frac{ds\,dt}{\sqrt{\mathbb{E}[|X_t - X_s|^2]}} < +\infty \tag{11}$$

is sufficient for the local time of X to exist and to be a square integrable function of x. If (s,t) is away from the diagonal we write for $s < t$

$$\mathbb{E}[|B_h(t) - B_h(s)|^2] = \int_0^s (K_{h(t)}(t,u) - K_{h(s)}(s,u))^2du + \int_s^t K_{h(t)}(t,u)^2du$$

and deduce from (1) that the second term stays strictly positive as (s,t) varies in $[0, t-\varepsilon] \times [0,T]$. If (s,t) is close to the diagonal, say $0 \leq t - s < \varepsilon$, by Proposition 5,

$$\mathbb{E}[|B_h(t) - B_h(s)|^2] \sim c_{h(t)}^{-2}(t-s)^{2h(t)}$$

as $s \nearrow t$, and a direct calculation shows that (11) is satisfied. ■

Let us now derive a Tanaka-type formula for B_h. Since the last term in the Itô formula (9) is an integral with respect to the variance function R and since R is not in general increasing, but only of finite variation, the trajectorial representation of the local time is more delicate than for Brownian motion or for fractional Brownian motion. On the time intervals where R is (strictly) inceasing or decreasing, this formula gives in fact an occupation density $\widehat{L}$ related to L.

Theorem 12 *Suppose that h is continuously differentiable. Then, for all $a \in \mathbb{R}$,*

$$|B_h(t) - a| - |B_h(s) - a| = \int_s^t \operatorname{sign}(B_h(u) - a) dB_h(u) + \widehat{L}([s,t], a)$$

$P-$a.s., where $\widehat{L}([s,t],a) = \int_s^t R'(u) L(du, a)$ and L is the local time with respect to the Lebesgue measure of B_h. On the time intervals $[s,t]$ where R is strictly inceasing (resp. strictly decreasing), $\widehat{L}$ (resp. $-\widehat{L}$) is the (positive) occupation density of B_h with respect to the measure induced by R.

Remark 13
a) No information on the local time of B_h can be drawn from the above formula if $R'(u) = 0$ on the interval $[s,t]$. In fact,

$$R'(u) = \frac{d}{du}\mathbb{E}[B^2_{h(u)}(u)] = \frac{d}{du}(u^{2h(u)}) = 2\left(h'(u)\log u + \frac{h(u)}{u}\right) u^{2h(u)},$$

and $R'(u) = 0$ if $h(u) = 1/\log u \in (1/2, 1)$ on an interval (s,t). In this case $\widehat{L}([s,t],a) = 0$ for all a.
b) Tanaka formulas for fractional Brownian motion have been shown by several authors. We refer to the survey [8] for references.

Proof. For $\varepsilon > 0$ let $p_\varepsilon(x) = (2\pi\varepsilon)^{-1/2}\exp(-x^2/(2\varepsilon))$. We apply the Itô formula of the previous section to the process

$$F_\varepsilon(x) = \int_0^t F'_\varepsilon(y) dy$$

where

$$F'_\varepsilon(x) = 2\int_{-\infty}^x p_\varepsilon(y) dy - 1.$$

Then $F'_\varepsilon(x)$ converges to $\operatorname{sign}(x)$ and $F_\varepsilon(x)$ converges to $|x|$ as $\varepsilon \to 0$. By (9), for $\varepsilon > 0$ fixed,

$$\begin{aligned} F_\varepsilon(B_h(t) - a) &= F_\varepsilon(-a) + \int_0^t \int_s^t F'_\varepsilon(B_h(r) - a) K(dr, s) \delta W_s \\ &\quad + \int_0^t p_\epsilon(B_h(s) - a) dR(s). \end{aligned} \tag{12}$$

Notice that by ($\mathbb{K}4$) the process $\{F'_\varepsilon(B_h(r) - a), r \in [0,t]\}$ belongs to $L^2(\Omega, \mathcal{H}_K)$ and belongs therefore to $\operatorname{Dom}(\delta^{B_h})$. Or, equivalently, the process $\{\int_s^t F'_\varepsilon(B_h(r) - a) K_h(dr, s), s \in [0,t]\}$ belongs to $\operatorname{Dom}(\delta^W)$. Clearly $F_\varepsilon(B_h(t) - a)$ converges to $|B_h(t) - a|$ in $L^2(\Omega)$ and

$F_\epsilon(-a)$ converges to $|a|$ if $\varepsilon \to 0$. Moreover, the process $\{F'_\varepsilon(B_h(r)-a), r \in [0,t]\}$ converges, as $\varepsilon \to 0$, to $\{\text{sign}(B_h(r)-a), r \in [0,t]\}$ in $L^2(\Omega, \mathcal{H}_K)$. In fact, by ($\mathbb{K}4$),

$$\mathbb{E}\left[\int_0^t \left(\int_s^t |F'_\varepsilon(B_h(r)-a) - \text{sign}(B_h(r)-a)||K_h|(dr,s)\right)^2 ds\right]$$

is upper bounded independently of ε, and we may apply the dominated convergence theorem (Lemma 1 of [9]). Therefore the last term in (12) converges in $L^2(\Omega)$. Let us denote the limit by Λ_t^a. Therefore, for any continous function f with compact support in $\mathbb{R}$,

$$\int \left(\int_0^t p_\varepsilon(B_h(s)) - a) dR(s)\right) f(a) da \tag{13}$$

converges in $L^1(\Omega)$ to $\int \Lambda_t^a f(a) da$. In fact, the dominated convergence theorem applies, because

$$\int_0^t \mathbb{E}[p_\varepsilon(B_h(s)) - a)] dR(s) = \int_0^t p_{R(s)+\varepsilon}(a) dR(s) \leqq \int_0^t s^{-b} R'(s) ds < \infty.$$

But (13) converges also to $\int_0^t f(B_h(s)) dR(s) = \int_0^t f(B_h(s)) R'(s) ds$, where we use the fact that R is differentiable if h is differentiable. Hence

$$\int_0^t f(B_h(s)) R'(s) ds = \int \Lambda_t^a f(a) da.$$

By the occupation density formula (10) applied to $g(s,x) = f(x)R'(s)$ we get $\Lambda_t^a = \int_0^t R'(s) L(ds,a) = \widehat{L}(t,a)$ for $\lambda-$a.e. a, $P-$a.s. We can extend to all $a \in \mathbb{R}$ since there exists a jointly continous version of L and therefore of $\widehat{L}$, as will be shown next by Berman's methods which are independent of the Tanaka formula. ■

We state now regularity properties in time and space of the trajectories of L. The regularity properties of $\widehat{L}$ follow easily since $\widehat{L}(t,x) = \int_0^t R'(s) L(ds,x)$. In order to show the existence of a jointly continuous version of L, we recall the hypotheses introduced in [7] and show that they are satisfied for B_h. We recall them in terms of any real valued separable random process $\{X(t), t \in [0,T]\}$ with Borel sample functions.

- **Hypothesis** ($\mathbb{A}$). There exist numbers $\rho_0 > 0$ and $H \in (0,1)$ and a positive function $\psi \in L^1(\mathbb{R})$ such that for all $\lambda \in \mathbb{R}$ and $t,s \in [0,T]$, $0 < |t-s| < \rho_0$ we have
$$\left|\mathbb{E}\left[\exp\left(i\lambda \frac{X(t)-X(s)}{|t-s|^H}\right)\right]\right| \le \psi(\lambda).$$

- **Hypothesis** ($\mathbb{A}_m$). There exist positive constants δ and c (both eventually depending on $m \ge 2$) such that for all $t_1, t_2, \cdots, t_m$ with $0 =: t_0 < t_1 < \cdots < t_m \le T$ and $|t_m - t_1| < \delta$ we have
$$\left|\mathbb{E}\left[\exp\left(i\sum_{j=1}^m u_j(X(t_j) - X(t_{j-1}))\right)\right]\right| \le \prod_{j=1}^m |\mathbb{E}[\exp(icu_j(X(t_j) - X(t_{j-1})))]|$$
for all $u_1, u_2, \cdots, u_m \in \mathbb{R}$.

Hypothesis ($\mathbb{A}$) is evidently satisfied for selfsimilar processes with stationary increments: here $\psi(\lambda) = |\mathbb{E}[e^{i\lambda X(1)}]|$. Hypothesis ($\mathbb{A}$) is also closely related to asymptotic selfsimilarity, and it holds in fact for a fairly large class of processes (see Proposition 15).

If X has independent increments, ($\mathbb{A}_m$) is trivially true for all $m \geq 2$. When the left and the right side of the inequality is applied to the characteristic function of a Gaussian process X, we get the condition which is known in the litterature under the name of *local nondeterminism* (LND). Loosely speaking, LND says how much dependence is allowed for the increments of the process if the local time should have certain regularity properties. As a general rule, the trajectories of local time get more regular if the trajectories of the process get less regular.

Proposition 14 *For all $T > 0$, the process $\{B_h(t), t \in [0,T]\}$ satisfies Hypotheses* ($\mathbb{A}$) *and* ($\mathbb{A}_m$) *for all $m \geq 2$ with $H = \max_{0 \leq t \leq T} h(t)$.*

Proof. Let us start by showing ($\mathbb{A}$). By similar calculations as in the proof of Proposition 5, for every $t \in [0,T]$

$$\left| \mathbb{E}[\varepsilon^{-2h(t)}(B_h(t+\varepsilon) - B_h(t))^2] - \frac{1}{c_{h(t)}^2} \right| \leq \varepsilon^{\beta - \sup_{[0,T]} h} \int_0^{T+1} (\Phi_{T+1}(s))^2 ds.$$

We can now conclude by applying Proposition 15.

Now we prove ($\mathbb{A}_m$) for all $m \geq 2$. We show that B_h satisfies the LND property as it has been introduced by S. Berman for Gaussian processes. For simplicity we write B instead of B_h in this proof. Let $t_1 < t_2 < \cdots < t_m$, and let $\mathcal{V}_m$ be the relative conditioning error given by

$$\mathcal{V}_m = \frac{\mathrm{Var}[B(t_m) - B(t_{m-1}) | B(t_1), \cdots, B(t_{m-1})]}{\mathrm{Var}[B(t_m) - B(t_{m-1})]}.$$

The Gaussian process is said to be LND if

$$\liminf_{\substack{c \searrow 0^+ \\ 0 < t_m - t_1 \leq c}} \mathcal{V}_m > 0. \tag{14}$$

This condition means that a small increment of the process is not completely predictable on the basis of a finite number of observations from the immediate past. For Gaussian processes S. Berman [4] has proved that (14) implies ($\mathbb{A}_m$). More precisely he has shown that if X satisfies (14) for all $m \geq 2$, then there exist constants $C_m > 0$ and $\delta_m > 0$ such that, for all $u_1, u_2, \cdots, u_m \in \mathbb{R}$,

$$\mathrm{Var}\left[\sum_{j=1}^m u_j [B(t_j) - B(t_{j-1})] \right] \geq C_m \sum_{j=1}^m u_j^2 \mathrm{Var}\left[B(t_j) - B(t_{j-1})\right],$$

where $t_0 = 0$ and $t_1 < \cdots < t_m$ are different and lie in an interval of length at most δ_m. This implies ($\mathbb{A}_m$). In order to prove that B verifies (14), fix $m \geq 2$ and let $t < t_1 < t_2 < \cdots < t_m < t + \delta t$. By Proposition 2

$$\mathrm{Var}[B(t_m) - B(t_{m-1})] \leq 2\mathrm{Var}[B(t_m) - B(t)] + 2\mathrm{Var}[B(t_{m-1}) - B(t)] \leq C_{h(t)} \delta^{2h(t)},$$

where $C_{h(t)}$ is a constant depending on t. Therefore

$$\lim_{\delta\to 0}\frac{\mathrm{Var}[B(t_m)-B(t_{m-1})|B(t_1),\cdots,B(t_{m-1})]}{\mathrm{Var}[B(t_m)-B(t_{m-1})]} \geq \lim_{\delta\to 0}\frac{\mathrm{Var}[B(t_m)|B(t_1),\cdots,B(t_{m-1})]}{C_{h(t)}\delta^{2h(t)}}.$$

Moreover, by adding $B(t)$ to the conditional set,

$$\begin{aligned}\frac{\mathrm{Var}[B(t_m)|B(t_1),\cdots,B(t_{m-1})]}{\delta^{2h(t)}} &\geq \frac{\mathrm{Var}[B(t_m)|B(t),B(t_1),\cdots,B(t_{m-1})]}{\delta^{2h(t)}} \\ &= \mathrm{Var}\left[\frac{B(t_m)-B(t)}{\delta^{h(t)}}\middle|B(t),\frac{B(t_1)-B(t)}{\delta^{h(t)}},\cdots,\frac{B(t_{m-1})-B(t)}{\delta^{h(t)}}\right]\end{aligned}$$

since

$$\sigma\{B(t),B(t_1)-B(t),\cdots,B(t_{m-1})-B(t)\} = \sigma\left\{B(t),\frac{B(t_1)-B(t)}{\delta^{h(t)}},\cdots,\frac{B(t_{m-1})-B(t)}{\delta^{h(t)}}\right\}.$$

Let now $u_1,\cdots,u_m$ be defined by $t_i-t=u_i\delta$, $i=1,2,\cdots,m$. Then $0<u_i<t$, and when $\delta\to 0$ the $t_i=t_i^\delta$ are chosen in such a way that the u_i do not change. Therefore

$$\begin{aligned}&\lim_{\delta\to 0}\mathrm{Var}\left[\frac{B(t+\delta u_m)-B(t)}{\delta^{h(t)}}\middle|B(t),\frac{B(t+\delta u_1)-B(t)}{\delta^{h(t)}},\cdots,\frac{B(t+\delta u_{m-1})-B(t)}{\delta^{h(t)}}\right] \\ &\qquad = \lim_{\delta\to 0}\mathrm{Var}\left[Y_{t,\delta}(u_m)\middle|B(t),Y_{t,\delta}(u_1),\cdots,Y_{t,\delta}(u_{m-1})\right],\end{aligned}$$

where $Y_{t,\delta}(u_i)=(B(t+\delta u_i)-B(t))/\,\delta^{h(t)}$. Moreover,

$$\mathrm{Var}[Y_{t,\delta}(u_m)|B(t),Y_{t,\delta}(u_1),\cdots,Y_{t,\delta}(u_{m-1})] = \frac{\det\mathrm{Cov}[B(t),Y_{t,\delta}(u_1),\cdots,Y_{t,\delta}(u_m)]}{\det\mathrm{Cov}[B(t),Y_{t,\delta}(u_1),\cdots,Y_{t,\delta}(u_{m-1})]}.$$

By the local selfsimilarity of B (Proposition 5) $Y_{t,\delta}$ converges weakly to a fractional Brownian motion $\tilde{B}_{h(t)}$ with Hurst parameter $h(t)$ (recall that t is fixed). Consequently the fraction above converges to

$$\frac{\det\mathrm{Cov}[B(t),\tilde{B}_{h(t)}(u_1),\cdots,\tilde{B}_{h(t)}(u_m)]}{\det\mathrm{Cov}[B(t),\tilde{B}_{h(t)}(u_1),\cdots,\tilde{B}_{h(t)}(u_{m-1})]}.$$

Therefore

$$\begin{aligned}&\lim_{\delta\to 0}\mathrm{Var}[Y_{t,\delta}(u_m)|B(t),Y_{t,\delta}(u_1),\cdots,Y_{t,\delta}(u_{m-1})) \\ &\qquad = \mathrm{Var}[\tilde{B}_{h(t)}(u_m)|\tilde{B}_{h(t)}(t),\tilde{B}_{h(t)}(u_1),\cdots,\tilde{B}_{h(t)}(u_{m-1})) \\ &\qquad \geq C_{h(t)}[(u_m-u_{m-1})\wedge(t-u_m)]^{2h(t)},\end{aligned}$$

where the last inequality follows from Lemma 7.1 of [13]. The last term is strictly positive since the $0 < u_1 < \cdots < u_m < t$. ■

The beginning of the proof of Proposition 14 shows that the hypothesis $(\mathbb{A})$ holds for a much larger class of processes. In fact we have only used the assumption of the following proposition.

Proposition 15 *Let $\{X_t, t \in [0,T]\}$be a centered Gaussian process. Suppose that for some positive continuous functions $f : [0,T] \to (0,1)$ and $g : [0,T] \to (0,\infty)$*

$$\mathbb{E}[\varepsilon^{-2f(t)}(X(t+\varepsilon) - X(t))^2] \underset{\varepsilon \to 0}{\longrightarrow} g(t) \tag{15}$$

uniformly in t. Then Hypothesis $(\mathbb{A})$ holds.

Proof. Let us fix $H > \sup f =: f^*$. Because X is Gaussian and centered, we have

$$\mathbb{E}\left[\exp\left(i\lambda \frac{X(t+\varepsilon) - X(t)}{\varepsilon^H}\right)\right] = \exp\left(-\frac{\lambda^2}{2}\mathbb{E}\left[\left(\frac{X(t+\varepsilon) - X(t)}{\varepsilon^H}\right)^2\right]\right). \tag{16}$$

Because of (15), there exists ε_0 such that for every ε satisfying $|\varepsilon| < \varepsilon_0$ and for every t we have

$$\mathbb{E}\left[\left(\frac{X(t+\varepsilon) - X(t)}{\varepsilon^{f(t)}}\right)^2\right] \geq \frac{c}{2} \tag{17}$$

where $c = \inf_{[0,T]} g$. Besides,

$$\varepsilon^{2f(t)-2H} \geq \varepsilon^{2f^*-2H} \geq 1, \tag{18}$$

and thus

$$\mathbb{E}\left[\left(\frac{X(t+\varepsilon) - X(t)}{\varepsilon^H}\right)^2\right] = \varepsilon^{2f(t)-2H}\mathbb{E}\left[\left(\frac{X(t+\varepsilon) - X(t)}{\varepsilon^{f(t)}}\right)^2\right] \geq \frac{c}{2}. \tag{19}$$

Then, combining (16) and (19) we get for every λ, and t and s satisfying $|t-s| < \varepsilon_0$,

$$\left|\mathbb{E}\left[\exp\left(i\lambda \frac{X(t+\varepsilon) - X(t)}{\varepsilon^H}\right)\right]\right| \leq \exp\left(-\frac{\lambda^2 c}{4}\right).$$

Then we choose $\psi(\lambda) = \exp\left(-\lambda^2 c/4\right)$ to conclude the proof. ■

Let us now state a regularity result for the trajectories of L. The following theorem says that $L(t,x)$ is Hölder continuous in t of order $1-H$ and Hölder continuous in x of order $\frac{1-H}{2H}$, where H is the constant appearing in $(\mathbb{A})$. For the proofs we refer to [7], where it is shown that these regularities hold for any process starting at zero and satisfying $(\mathbb{A})$ and $(\mathbb{A}_m)$ for all $m \geq 2$.

Theorem 16 *Suppose that ψ in $(\mathbb{A})$ satisfies*

$$\int_{|u|\geq 1} |u|^{(1-H)/H}\psi(u)du < \infty.$$

Then $\{B_h(t), t \in [0,T]\}$ has a jointly continuous local time $\{L(t,x), (t,x) \in [0,T] \times \mathbb{R}\}$. Moreover, for any compact set $K \subset \mathbb{R}$ and any interval $I \subset [0,T]$ with length less than ρ_0 (the constant appearing in $(\mathbb{A})$),
(i) if $0 < \xi < (1-H)/2H$, then $|L(I,x) - L(I,y)| \leq \eta|x-y|^\xi$ for all $x, y \in K$
(ii) if $0 < \delta < 1-H$, then $\sup_{x\in K} L(I,x) \leq \eta|I|^\delta$, where η is a random variable, almost surely positive and finite and $|I|$ is the length of I.

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